Singular blocks of parabolic category $\mathcal{O}$ and finite W-algebras

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Abstract. We show that each integral infinitesimal block of parabolic category $\mathcal{O}$ (including singular ones) for a semi-simple Lie algebra can be realized as a full subcategory of a "thick" category $\mathcal{O}$ over a finite W-algebra for the same Lie algebra.

The nilpotent used to construct this finite W-algebra is determined by the central character of the block, and the subcategory taken is that killed by a particular two-sided ideal depending on the original parabolic. The equivalences in question are induced by those defined by Miličić-Soergel and Losev.

We also give a proof of a result of some independent interest: the singular blocks of parabolic category $\mathcal{O}$ can be geometrically realized as "partial Whittaker sheaves" on partial flag varieties.

In this note, we aim to clarify the relationship between the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand [BGG76] and the category $\mathcal{O}$ over the finite W-algebra defined by Brundan, Goodwin and Kleshchev [BGK08]. We show that:

Theorem *Any integral infinitesimal block of parabolic category $\mathcal{O}$ for a semi-simple Lie algebra $\mathfrak{g}$ is equivalent to a category of representations over a finite W-algebra for $\mathfrak{g}$ corresponding to a nilpotent (depending on the choice of central character).*

Recall that an **infinitesimal block** of a category of $U(\mathfrak{g})$-modules is defined to be the subcategory generated by the simple modules on which the center of $U(\mathfrak{g})$ acts by a fixed central character.

For a more precise statement of this Theorem, see Corollary 6. The equivalences refered to above are obtained by restricting to subcategories equivalences defined in previous work by Miličić-Soergel [MS97] on representations of Lie algebras and Losev [Losa] on finite W-algebras.

The category of W-algebra modules in question has a flavor similar to the category $\mathcal{O}$ for the W-algebra already defined by Brundan, Goodwin and Kleshchev [BGK08], but it is subtly different.

In order to explain this difference, consider the case of our parabolic being the Borel itself, the original definition of category $\mathcal{O}$ considered in [BGG76]. In this case, we show that a singular block of category $\mathcal{O}$ is equivalent to the category of modules over a W-algebra for the same Lie algebra (with nilpotent depending on the singular block) generated by "Verma modules" for the W-algebra defined by Brundan, Goodwin and Kleshchev [BGK08, §4.2], and which furthermore have trivial central character. We will refer to this category as category $\mathcal{O}'$ for the finite W-algebra.

This category has two subtle differences from the category $\mathcal{O}$'s defined in [BGG76] for $U(\mathfrak{g})$ and in [BGK08] for the W-algebra:

- The category $\mathcal{O}'$ only contains modules where the center acts semi-simply.
- We have no analogue of the condition that the Cartan act semi-simply, which is imposed in [BGG76] (an analogue of this semi-simplicity is also imposed in [BGK08]).

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If instead of the Borel, we begin with a block of parabolic category $\mathcal{O}$ for some larger parabolic, we must further impose the condition that these modules are killed by a particular two-sided ideal.

The proof of the theorem above is organized into two sections. Section 1 covers preparatory results on category $\mathcal{O}$ and Harish-Chandra bimodules; it is likely that many of the results therein are familiar to experts, but they do not seem to have been written down together in the necessary form anywhere in the literature. Section 2 describes the relationship of these results to the representation theory of the finite W-algebra, and completes the proof.

Finally, in Section 3, we also discuss briefly the relationship of these results to geometry. In particular, we give a proof of a result which has circulated as a folk theorem: the singular integral infinitesimal blocks of parabolic category $\mathcal{O}$ can be geometrically realized as (N, f)-equivariant (the so-called "partial Whittaker") D-modules or perverse sheaves on G/Q. This is a classical theorem when f is generic (going back to a paper of Kostant [**?**]), and the $Q = B$ case follows from work of Bezrukavnikov and Yun [BY, Theorem 4.4.1] but we know of no proof in the literature of the general case.

In a forthcoming paper [BLPW], the author, Braden, Licata and Proudfoot will discuss the relationship to geometry more fully and relate this isomorphism to the geometry of strata in a Slodowy slice and to our conjectures on symplectic duality between such strata.

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1 Lie theory

First, we fix notation. Let $\mathfrak{g} \supset \mathfrak{q} \supset \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a semi-simple complex Lie algebra, a parabolic, a Borel, and a chosen decomposition of $\mathfrak{b}$ as a Cartan and its nilpotent radical. Let $G \supset Q \supset B = HN$ be connected groups with these Lie algebras (which form of G we take is irrelevant to our purposes). Let Z denote the center of the universal enveloping algebra $U(\mathfrak{g})$.

Definition 1 We let $\widehat{\mathcal{O}}$ be the category of representations of $\mathfrak{g}$ where $\mathfrak{n}$ and Z act locally finitely.

This category has a natural infinitesimal block decomposition $\widehat{\mathcal{O}} = \bigoplus \widehat{\mathcal{O}}(\chi, f)$ where the sum runs over characters $\chi\colon Z \to \mathbb{C}$ and $f\colon \mathfrak{n} \to \mathbb{C}$ and $\widehat{\mathcal{O}}(\chi, f)$ is the subcategory where all the irreducible constituents of the restriction to $U(\mathfrak{n}) \otimes Z$ are $f \otimes \chi$. For any highest weight λ, we let χ_λ be the character of Z acting on the unique irreducible $\mathfrak{g}$-module of highest weight λ.

The category $\widehat{\mathcal{O}}$ contains two very natural subcategories:

- $\mathcal{O}$ is the subcategory of modules on which $\mathfrak{h}$ acts semi-simply.
- $\mathcal{O}'$ is the subcategory of modules on which Z acts semi-simply.

These have the same infinitesimal block decomposition as above. We note that $\mathcal{O}$ is the most commonly studied of these categories, and $\bigoplus_\chi \mathcal{O}(\chi, 0)$ in our notation is precisely the category $\mathcal{O}$

originally defined by Bernstein-Gelfand-Gelfand [BGG76], and $\mathcal{O}(\chi, 0)$ its infinitesimal block for the central character χ.

It's worth noting that for non-integral χ, the infinitesimal blocks $\mathcal{O}(\chi, 0)$ are not necessarily blocks in the abstract sense; they may have further decompositions as the sum of orthogonal subcategories. For several infinitesimal blocks of categories we consider, the abstract block decompositions can actually be quite subtle and complicated.

We will also wish to consider the parabolic version of category $\mathcal{O}$. Let $\mathcal{O}^{\mathfrak{q}}(\chi, 0)$ be the subcategory of $\mathcal{O}(\chi, 0)$ consisting of modules also locally finite for the action of $\mathfrak{q}$. One particular special object in $\mathcal{O}^{\mathfrak{q}}(\chi_0, 0)$ is the dominant parabolic Verma module $M^{\mathfrak{q}} = \operatorname{Ind}_{U(\mathfrak{q})}^{U(\mathfrak{g})} \mathbb{1}$, where $\mathbb{1}$ is the trivial representation of $\mathfrak{q}$.

For the entirety of the paper, we will always use f and μ to denote a Lie algebra map $f \colon \mathfrak{n} \to \mathbb{C}$ and a weight μ of $\mathfrak{g}$ such that if Δ_f be the set of simple roots α whose root spaces are not killed by f, then $\alpha^\vee(\mu) = -1$ for all $\alpha \in \Delta_f$ and $\alpha^\vee(\mu) \in \mathbb{Z} \setminus \{-1\}$ for all $\alpha \notin \Delta_f$. In particular, note that $f = 0$ and $\mu = 0$ are compatible in the sense above; this will be an important special case for us.

Let $\mathfrak{p}_f$ be the parabolic generated by the standard Borel $\mathfrak{b}$ and the negative root spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_f$.

The starting point for us is an equivalence of categories $\Phi_\mu \colon \widehat{\mathcal{O}}(\chi_\mu, 0) \to \widehat{\mathcal{O}}(\chi_0, f)$ constructed by Miličić and Soergel [MS97, Theorem 5.1]. This functor makes sense for any integral value of μ for an appropriate value of f; one case of special interest is the functor $\Phi_0 \colon \widehat{\mathcal{O}}(\chi_0, 0) \to \widehat{\mathcal{O}}(\chi_0, 0)$. This functor was considered in earlier work of Soergel [**?**] to construct an equivalence between $\mathcal{O}(\chi_0, 0)$ and $\mathcal{O}'(\chi_0, 0)$. Our main result of this section will be to understand the effect of this functor on subcategories of interest to us.

We let $I_{\mathfrak{q}} = \operatorname{Ann}_{U(\mathfrak{g})} \Phi_0(M^{\mathfrak{q}})$ and let $\mathcal{O}'_{\mathfrak{q}}(\chi_0, f)$ denote the subcategory of modules killed by $I_{\mathfrak{q}}$. It may look peculiar to the reader that we are using $M^{\mathfrak{q}}$, a Verma module of central character χ_0 to define an ideal, and then using this ideal to define a category related to $\mathcal{O}(\chi_\mu, 0)$; this creates no problems, since all the target categories for Φ_* have central character χ_0. We note that $I_{\mathfrak{q}} \cap Z = \ker \chi_0$, since $\Phi_0(M^{\mathfrak{q}})$ is a quotient of a Verma module. Thus any object of $\widehat{\mathcal{O}}(\chi_0, f)$ killed by $I_{\mathfrak{q}}$ is automatically a semi-simple Z module, and therefore in $\mathcal{O}'(\chi_0, f)$.

Proposition 2 *The subalgebra $\mathfrak{q}$ acts locally finitely and $\mathfrak{h}$ acts semi-simply on a module $X \in \widehat{\mathcal{O}}(\chi_\mu, 0)$ if and only if $\operatorname{Ann}(\Phi_\mu(X)) \supset I_{\mathfrak{q}}$; that is, we have an induced equivalence $\Phi_\mu \colon \mathcal{O}^{\mathfrak{q}}(\chi_\mu, 0) \to \mathcal{O}'_{\mathfrak{q}}(\chi_0, f)$.*

In particular, setting $\mathfrak{q} = \mathfrak{b}$, we have that $\mathfrak{h}$ acts semi-simply on $X \in \widehat{\mathcal{O}}(\chi_\mu, 0)$ if and only if Z acts semi-simply on $\Phi_\mu(X)$; that is, we have an induced equivalence $\Phi_\mu \colon \mathcal{O}(\chi_\mu, 0) \to \mathcal{O}'(\chi_0, f)$.

Most of the ideas necessary for the proof of this fact are already in the literature, which we now collect into the

Lemma 3 *For each object $X \in \widehat{\mathcal{O}}(\chi_\mu, 0)$ and each projective functor $\mathcal{P} \colon \widehat{\mathcal{O}}(\chi_{\mu'}, 0) \to \widehat{\mathcal{O}}(\chi_\mu, 0)$, we have an inclusion $\operatorname{Ann}_{U(\mathfrak{g})}(\Phi_\mu(\mathcal{P}X)) \supset \operatorname{Ann}_{U(\mathfrak{g})}(\Phi_{\mu'}(X))$.*

If L_1 and L_2 are simple, then this statement has a converse; L_2 appears as a composition factor in $\mathcal{P}L_1$ for some projective functor if and only if $\operatorname{Ann}_{U(\mathfrak{g})}(\Phi_\mu L_1) \subset \operatorname{Ann}_{U(\mathfrak{g})}(\Phi_{\mu'} L_2)$.

Proof: The Miličić-Soergel result depends on equivalences of categories

$$ {}_{\chi_\mu}\mathcal{H}_{\chi_0} \xrightarrow{\varphi_1} \widehat{\mathcal{O}}(\chi_\mu, 0) \qquad\qquad {}_{\chi_0}\mathcal{H}_{\chi_\mu} \xrightarrow{\varphi_2} \widehat{\mathcal{O}}(\chi_0, f) $$

with certain blocks of the category $\mathcal{H}$ of Harish-Chandra bimodules, defined by fixing the generalized character of the left and right Z-actions. The equivalence Φ_μ is given by composing these with the "flip equivalence" $\mathcal{F} \colon {}_{\chi_\mu}\mathcal{H}_{\chi_0} \to {}_{\chi_0}\mathcal{H}_{\chi_\mu}$, so we have $\Phi_\mu = \varphi_2 \mathcal{F} \varphi_1^{-1}$.

If we were being very careful, we would include μ and f in the notation, since this equivalence exists for any compatible choice, but for simplicity, we leave them out.

The functors φ_i are both of the form

$$\varphi_i(Y) = \varprojlim Y \otimes_{U(\mathfrak{g})} L_i^n$$

for an inverse system of representations L_i^n for $i = 1, 2$, and $n \geq 0$. Thus, for any $U(\mathfrak{g})$-bimodule D, we have $\varphi_i(D \otimes_{U(\mathfrak{g})} Y) \cong D \otimes_{U(\mathfrak{g})} \varphi_i(Y)$.

In particular, for any two-sided ideal I, we have $\varphi_i(IY) \cong I\varphi_i(Y)$. Also, for any projective functor $\mathcal{P} : \widehat{\mathcal{O}}(\chi_{\mu'}, 0) \to \widehat{\mathcal{O}}(\chi_\mu, 0)$ where we let $\mathcal{P}_L$ denote the functor on $U(\mathfrak{g})$-bimodules given by acting with the projective functor $\mathcal{P}$ on the left, then by the same principle, we have $\mathcal{P}\varphi_1 = \varphi_1\mathcal{P}_L$.

One important consequence of this fact is that the left annihilator of a bimodule Y coincides with the annihilator of $\varphi_i(Y)$. Thus, $\varphi_1^{-1}(X)$ is a Harish-Chandra bimodule whose left annihilator is that of X and whose right annihilator is that of $\Phi_\mu(X)$. Thus, we have

$$\mathrm{Ann}(\Phi_\mu(\mathcal{P}X)) = \mathrm{RAnn}(\mathcal{P}_L\varphi_1^{-1}(X)) \supset \mathrm{RAnn}(\varphi_1^{-1}(X)) = \mathrm{Ann}(\Phi_{\mu'}(X)).$$

The first part of the result follows.

Now, we prove the second statement (the partial converse). Consider the Harish-Chandra bimodules $\varphi_1^{-1}(L_1)$ and $\varphi_1^{-1}(L_2)$. By reversing left and right in [Vog80, Theorem 3.2], we have

$$\mathrm{Ann}(\Phi_\mu L_1) = \mathrm{RAnn}(\varphi_1^{-1}(L_1)) \subset \mathrm{RAnn}(\varphi_1^{-1}(L_2)) = \mathrm{Ann}(\Phi_{\mu'} L_2)$$

if and only if there is a projective functor $\mathcal{P}$ such that $\varphi_1^{-1}(L_2)$ is a composition factor in $\mathcal{P}_L\varphi_1^{-1}(L_1)$. Applying the equivalence φ_1, this is true if and only if L_2 is a composition factor of $\mathcal{P}L_1$. $\square$

Proof of Proposition 2: If X is in $\mathcal{O}^{\mathfrak{q}}(\chi_\mu, 0)$, then there is a projective functor $\mathcal{P}\colon \mathcal{O}^{\mathfrak{q}}(\chi_0, 0) \to \mathcal{O}^{\mathfrak{q}}(\chi_\mu, 0)$ and a surjection $\mathcal{P}M^{\mathfrak{q}} \to X$, (this is an old result; it follows, for example, from [Kho05, Proposition 22]) and thus, by Lemma 3, we have inclusions

$$\mathrm{Ann}(\Phi_0(M^{\mathfrak{q}})) \subset \mathrm{Ann}(\Phi_\mu(\mathcal{P}M^{\mathfrak{q}})) \subset \mathrm{Ann}(\Phi_\mu(X)).$$

Assume $\Phi_\mu(X)$ is killed by $I_{\mathfrak{q}}$. By passing to composition factors, we may assume that X is simple. Let J_i be the primitive ideals killing the composition factors L_i of $M^{\mathfrak{q}}$, in the order induced by a Jordan-Hölder series. Then $J_1 \cdots J_m \subset I_{\mathfrak{q}} \subset \mathrm{Ann}(\Phi_\mu(X))$. Since $\mathrm{Ann}(\Phi_\mu(X))$ is prime, we have $J_i \subset \mathrm{Ann}(\Phi_\mu(X))$ for some i. Thus, by the second part of Lemma 3, X appears as a composition factor of $\mathcal{P}L_i$. Since L_i is $\mathfrak{q}$-locally finite, so is $\mathcal{P}L_i$, and thus X.

Finally, we establish the relationship between $\mathfrak{h}$-semi-simplicity and Z-semi-simplicity. By [MS97, Theorem 5.3] in the case where $n = 1$, we have that a bimodule $D \in {}_{\chi_\mu}\mathcal{H}_{\chi_0}$ is semi-simple as a right Z-module if and only if $\varphi_1(D)$ is a module on which the Cartan acts semi-simply, that is, a weight module. Thus, X is a weight module if and only if $\varphi_1^{-1}(X)$ is semi-simple as a right bimodule if and only if $\mathcal{F}\varphi_1^{-1}(X)$ is semi-simple as a left Z-module if and only if $\Phi_\mu(X)$ is semi-simple as a Z-module (applying the commutation of ideals past φ_2 again). $\square$

2 Finite W-algebras

Now, we will connect the previous section, which only dealt with representations of Lie algebras, to the study of finite W-algebras.

The finite W-algebra $\mathcal{W}_e$ is a infinite-dimensional associative algebra constructed from the data of $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$; geometrically, it is obtained by taking a quantization of the Slodowy slice to the orbit through e. It is most easily defined as the endomorphism ring of the Gelfand-Graev representation $Q_{\Xi,\mathfrak{m}} = \Xi \otimes_{U(\mathfrak{m})} U(\mathfrak{g})$ for a particular nilpotent subalgebra $\mathfrak{m} \subset \mathfrak{n}$ depending on e, and a one-dimensional representation Ξ of $\mathfrak{m}$ constructed using e. For more on the general theory of W-algebras, see [Pre02, GG02].

The algebra $\mathcal{W}_e$ has a category of representations analogous to $\widehat{\mathcal{O}}$ above, originating in the work of Brundan, Goodwin and Kleshchev [BGK08]. This category is associated a *choice* of parabolic $\mathfrak{p}$ such that

- e is distinguished in the Levi $\mathfrak{l}$ of $\mathfrak{p}$ and
- $\mathfrak{p}$ contains a fixed maximal torus $\mathfrak{t}$ of the centralizer $C_{\mathfrak{g}}(e) = \{x \in \mathfrak{g} | [x, e] = 0\}$.

This choice of parabolic allows us to put an preorder on the weights of $\mathfrak{t}$ by declaring that $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a linear combination of weights of $\mathfrak{t}$ acting on $\mathfrak{p}$.

We have a natural inclusion $U(\mathfrak{t}) \hookrightarrow \mathcal{W}_e$, (described in [BGK08, Theorem 3.3]) and thus can analyze any $\mathcal{W}_e$ representation by its restriction to $\mathfrak{t}$.

Definition 4 Let $\widehat{\mathcal{O}}(\mathcal{W}_e, \mathfrak{p})$ be the category of modules X such that

- $\mathfrak{t}$ acts on X with finite dimensional generalized weight spaces.
- the weights which appear in X are contained in a finite union of sets of the form $\{\mu | \mu \leq \lambda\}$.

As before, $\mathcal{O}(\mathcal{W}_e, \mathfrak{p})$ denotes the subcategory on which $\mathfrak{t}$ acts semi-simply, and $\mathcal{O}'(\mathcal{W}_e, \mathfrak{p})$ denotes the subcategory on which $Z \cong Z(\mathcal{W}_e)$ acts semi-simply.

Alternatively, following [Losa], we could define an analogue of $\widehat{\mathcal{O}}$ as the category of $\mathcal{W}_e$-modules which are locally finite for the action of the non-negative weight spaces of an element $\xi \in C_{\mathfrak{g}}(e)$ acting by the adjoint action on $\mathcal{W}_e$. This will recover the BGK definition when $\mathfrak{p}$ is precisely the non-negative weight spaces for the adjoint action of ξ on $\mathfrak{g}$. In Losev's notation, $\widehat{\mathcal{O}}(\mathcal{W}_e, \mathfrak{p})$ is denoted $\mathcal{O}(\xi)$, and the category we and [BGK08] denote $\mathcal{O}(\mathcal{W}_e, \mathfrak{p})$, Losev would denote $\mathcal{O}^{\mathfrak{t}}(\xi)$ (leaving the e implicit).

There is an isomorphism $Z \cong Z(\mathcal{W}_e)$, typically referenced in the literature to a footnote to Question 5.1 in the paper of Premet [Pre07], where Premet gives an argument he ascribes to Ginzburg. This isomorphism allows us to identify central characters of $U(\mathfrak{g})$ and $\mathcal{W}_e$. We denote the subcategory of $\widehat{\mathcal{O}}(\mathcal{W}_e, \mathfrak{p})$ with generalized central character χ by $\widehat{\mathcal{O}}(\chi, \mathcal{W}_e, \mathfrak{p})$, and the subcategory where Z acts semi-simply (and thus by the character χ) by $\mathcal{O}'(\chi, \mathcal{W}_e, \mathfrak{p})$.

We will primarily interested in the case where after fixing a parabolic $\mathfrak{p} \supset \mathfrak{b}$ with Levi $\mathfrak{l}$, we take e to be the principal nilpotent in $\mathfrak{l}$ (for the Borel given by intersection with $\mathfrak{b}$), and we will assume we are in this situation. In type A, all nilpotents will appear this way up to conjugacy, since we can take $\mathfrak{p}$ to be block upper triangular matrices attached to the Jordan blocks of e; in other types, there are nilpotents that are not of this form.

As before, let $f : \mathfrak{n} \to \mathbb{C}$ be a fixed character of this nilpotent Lie algebra.

Proposition 5 ([Losa]) *If e is principal in the Levi of $\mathfrak{p}_f$, there is an equivalence*

$$\mathcal{L} : \widehat{\mathcal{O}}(\chi_0, \mathcal{W}_e, \mathfrak{p}_f) \to \widehat{\mathcal{O}}(\chi_0, f)$$

inducing equivalences

$$\mathcal{O}'(\chi_0, \mathcal{W}_e, \mathfrak{p}_f) \cong \mathcal{O}'(\chi_0, f) \qquad \mathcal{O}(\chi_0, \mathcal{W}_e, \mathfrak{p}_f) \cong \mathcal{O}(\chi_0, f)$$

Furthermore, $\mathcal{L}(X)$ is killed by $I_{\mathfrak{q}}$ if and only if X is killed by a certain two-sided ideal $(I_{\mathfrak{q}})_{\dagger}$ of the W-algebra. We denote the subcategory killed by $(I_{\mathfrak{q}})_{\dagger}$ by $\widehat{\mathcal{O}}_{\mathfrak{q}}(\chi_0, \mathcal{W}_e)$.

The ideal $(I_{\mathfrak{q}})_{\dagger}$ arises from a pair of maps: $I \mapsto I^{\dagger}$ sends two-sided ideals in the W-algebra to two-sided ideals of $U(\mathfrak{g})$ and $J \mapsto J_{\dagger}$ goes the opposite direction; these both are defined by Losev in [Losb, §3.4]. In a forthcoming paper [BLPW], the author, Braden, Licata and Proudfoot will show that this ideal $(I_{\mathfrak{q}})_{\dagger}$ is primitive, and itself has a geometric interpretation.

Proof: The only part of the equivalences not stated directly in [Losa, Theorem 4.1] is the equivalence of most interest for us, that of $\mathcal{O}'(\chi_0, \mathcal{W}_e, \mathfrak{p}_f) \cong \mathcal{O}'(\chi_0, f)$.

It follows from [Losa, Theorem 4.1 (1)] and [Losb, Theorem 1.2.2(iii)] that

$(*)$: the center acts semi-simply on $X \in \widehat{\mathcal{O}}(\chi_0, \mathcal{W}_e, \mathfrak{p}_f)$ if and only if it acts semi-simply on $\mathcal{L}(X)$.

The "only if" portion of $(*)$ shows that $\mathcal{L}$ induces a fully faithful functor $\mathcal{L} : \mathcal{O}'(\chi_0, \mathcal{W}_e, \mathfrak{p}_f) \to \mathcal{O}'(\chi_0, f)$. The "if" portion of $(*)$ shows that the essential surjectivity of $\mathcal{L}$ implies the essential surjectivity of this restriction.

Now, fix a $U(\mathfrak{g})$-module X such that $\mathrm{Ann}(X) \supset I_{\mathfrak{q}}$, and consider $\mathrm{Ann}(\mathcal{L}^{-1}(X))$. We know by [Losb, Theorem 4.1(1)] that $\mathrm{Ann}(\mathcal{L}^{-1}(X))^{\dagger} = \mathrm{Ann}(X)$. It follows that $\mathrm{Ann}(\mathcal{L}^{-1}(X)) \supset \mathrm{Ann}(X)_{\dagger} \supset (I_{\mathfrak{q}})_{\dagger}$ since Losev's $\dagger$ maps both preserve inclusion (this follows from [Losb, Theorem 1.2.2(i)] for $(\cdot)_{\dagger}$ and is noted by Losev in [Losb, §3.4] for $(\cdot)^{\dagger}$).

On the hand, if $V \in \mathcal{O}'(\chi_0, \mathcal{W}_e, \mathfrak{p}_f)$ is killed by $(I_{\mathfrak{q}})_{\dagger}$. Then $\mathcal{L}(V)$ is a $U(\mathfrak{g})$-module killed by $((I_{\mathfrak{q}})_{\dagger})^{\dagger}$. By [Losb, Proposition 3.4.4], $((I_{\mathfrak{q}})_{\dagger})^{\dagger} \supset I_{\mathfrak{q}}$ and $\mathcal{L}(V)$ is killed by $I_{\mathfrak{q}}$. □

To recap, if μ is a weight of $\mathfrak{g}$ such that $\mu+\rho$ is dominant integral, and e is the principal nilpotent of the Levi in the parabolic $\mathfrak{p}_f$ corresponding to the stabilizer of $\mu+\rho$ under the usual action of W on weights, then combining Propositions 2 and 5, we have

Corollary 6 *There is an equivalence $\mathcal{O}^{\mathfrak{q}}(\chi_\mu, 0) \cong \mathcal{O}'_{\mathfrak{q}}(\chi_0, f) \cong \mathcal{O}'_{\mathfrak{q}}(\chi_0, \mathcal{W}_e, \mathfrak{p}_f)$.*

3 Relation to geometry

In order to give the reader some feeling for the "meaning" of the ideal $I_{\mathfrak{q}}$, we will briefly indicate its relationship to geometry. This relation will be drawn out much more fully in a forthcoming paper of the author, Braden, Licata, and Proudfoot.

We have the G-space G/Q, and as with any G-space, a map $\alpha\colon U(\mathfrak{g}) \to D(G/Q)$ to the ring of global differential operators on G/Q sending a Lie algebra element to the corresponding vector field.

Proposition 7 *The map α induces an isomorphism $D(G/Q) \cong U(\mathfrak{g})/I_{\mathfrak{q}}$. In particular, $\mathrm{gr}(U(\mathfrak{g})/I_{\mathfrak{q}}) = \mathbb{C}[T^*G/P]$, and $I_{\mathfrak{q}}$ is prime.*

Proof: Borho and Brylinski [BB82] show[2] that $D(G/Q) \cong U(\mathfrak{g})/I$ where I is the annihilator of a simple parabolic Verma module $L^{\mathfrak{q}}$ with highest weight $2\rho_Q - 2\rho$; this is the unique such with central character χ_0. Thus, we need only show that $I = I_{\mathfrak{q}}$.

We can understand the functor Φ_0 restricted to $\mathcal{O}(\chi_0, 0) \cap \mathcal{O}'(\chi_0, 0)$ as localizing a module to a B-equivariant D-module on G/B, then using the "flip" functor to think of this as a B-equivariant

[2]We thank the referee for pointing out this reference.

D-module on $B\backslash G$, and taking sections. In particular, the parabolic Verma module $M^{\mathfrak{q}}$ localizes to the !-extension of the structure sheaf $\mathcal{O}_{Qw_0B/B}$ as a D-module (don't confuse this with the use of the symbol $\mathcal{O}$ for a category; that will not appear in the rest of this proof), the structure sheaf of the dense Q-orbit, so $\Phi_0(M^{\mathfrak{q}})$ localizes to the !-extension of $\mathcal{O}_{Bw_0Q/B}$. Since the latter sheaf is smooth along the fibers of the natural map $G/B \to G/Q$, the pushforward of this sheaf to G/Q is the !-extension of $\mathcal{O}_{Bw_0Q/Q}$. In particular, $\Phi_0(M^{\mathfrak{q}})$ is the sections of a D-module on G/Q, and thus annihilated by I.

On the other hand, by the work of Irving [Irv85, §4.3], the socle of $M^{\mathfrak{q}}$ is in the same right cell as $L^{\mathfrak{q}}$. Thus, the socle of $\Phi_0(M^{\mathfrak{q}})$ is in the same left cell as $L^{\mathfrak{q}}$, and so these simple modules have the same annihilator. Since any ring element annihilating $\Phi_0(M^{\mathfrak{q}})$ annihilates its socle, we have that $I \supset I_{\mathfrak{q}}$, and equality of ideals follows.

This isomorphism, followed by the symbol map on differential operators induces an isomorphism $\mathrm{gr}(U(\mathfrak{g})/I_{\mathfrak{q}}) \cong \mathbb{C}[T^*G/Q]$. Since the associated graded of $U(\mathfrak{g})/I_{\mathfrak{q}}$ is the global function ring of an irreducible quasi-projective variety, the associated graded is an integral domain, and thus so is $U(\mathfrak{g})/I_{\mathfrak{q}}$. □

Interestingly, this establishes a conjecture which had appeared in an unpublished manuscript of Bezrukavnikov and Mirković, and which had been previously studied by the author and Stroppel.

Definition 8 We say that a D-module $\mathcal{M}$ on G/Q is $(\mathbf{N}, \mathbf{f})$**-equivariant** if the action of $\mathfrak{n}$ on $\mathcal{M}$ by $n \cdot m = \alpha_n m - f(n)m$ (where α_n is the vector field given by the infinitesimal action of n on G/Q) integrates to an N-equivariant structure on $\mathcal{M}$. Some authors use "strongly equivariant" for this form of equivariance.

There is also a version of (N, f)-equivariance for perverse sheaves on G/Q, though in the algebraic category this can only be made sense of using perverse sheaves on the characteristic p analogue of G/Q (for more on this approach, see [BBM04]). In this case, we let f' be a character $f' \colon N/\mathbb{F}_p \to \mathbb{G}_a/\mathbb{F}_p$ such that $\Delta_{f'} = \Delta_f$ and twist the equivariance condition by an Artin-Schreier sheaf pulled back from $\mathbb{G}_a/\mathbb{F}_p$ by f'. These are sometimes called "partial Whittaker" or "Whittavariant" sheaves.

Corollary 9 *The category $\mathcal{O}^{\mathfrak{q}}(\chi_\mu, 0)$ is equivalent to the category of (N, f)-equivariant D-modules on G/Q (and thus also to the category of (N, f')-equivariant perverse sheaves on G/Q, with all groups considered as algebraic groups over $\mathbb{F}_p$).*

Proof: By Beilinson-Bernstein [BB81], the sections functor $\mathcal{D}_{G/Q} - \mathsf{mod} \to U(\mathfrak{g})/I_{\mathfrak{q}} - \mathsf{mod}$ is an equivalence. If $\mathcal{M}$ is a (N, f)-equivariant D-module, then the action of $U(\mathfrak{n})$ on $\Gamma(\mathcal{M})$ is locally finite, with $\mathbb{C}_f$ being the only representation which appears in the composition series, so $M \in \mathcal{O}'_{\mathfrak{q}}(\chi_0, f)$. On the other hand, if $M \in \mathcal{O}'_{\mathfrak{q}}(\chi_0, f)$, then the f-twisted $\mathfrak{n}$-action via differential operators on the localization will integrate to an (N, f)-action. □

As we noted earlier, the $B = Q$ case of this can be derived from the "paradromic/Whittavariant" duality of [BY].

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